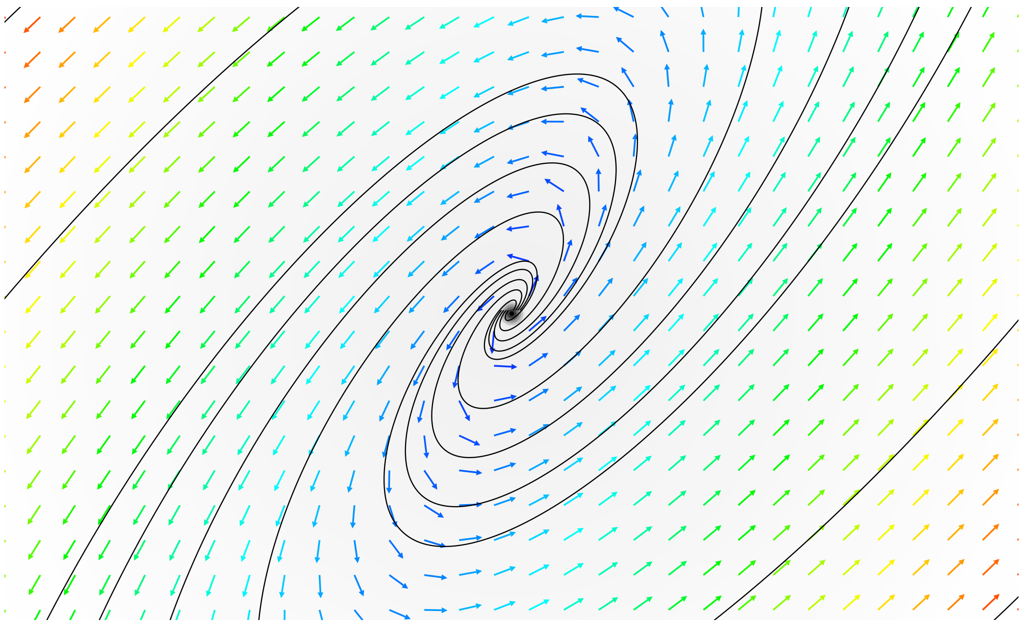
LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS



An exploration in the context of Matrix Theory

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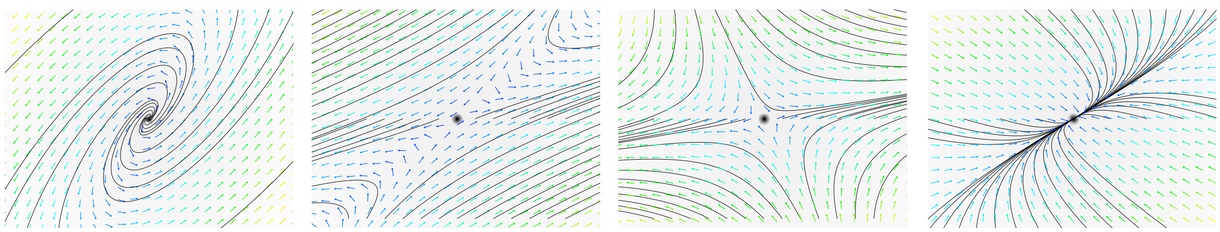
Introduction to Differential Equations

At the highest level, differential equations relate the change in one quantity with the change of another quantity. Many well-known basic physical properties are expressed in terms of differential equations, such as Newton's second law (***F*** = *m* ***a***). Outside of physics, there are also applications, such as predator-prey population relationships in a habitat.



Vector fields and phase portraits for various systems of linear differential equations

Though differential equations can be expressed in terms of many functions such as polynomials and transcendentals, we will primarily be interested in *linear* differential equations for the purposes of this exploration.

Notes on the number *e*

There are several convenient properties inherent to applications and instances of the number ***e***, Euler's number. This irrational number, approximately 2.71828..., arises in frequently in descriptions of natural functions and systems.

There are several methods for characterizing this number, but one of particular relevance to this exploration is the form of a sum of an infinite series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

This definition for e should look familiar after seeing the series definition for e^{At}:

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k = \frac{A^0}{0!} t^0 + \frac{A^1}{1!} t^1 + \frac{A^2}{2!} t^2 + \frac{A^3}{3!} t^3 + \frac{A^4}{4!} t^4 + \ldots$$

From this we can note that the first term in the summation will always be 1.

Patterns in e^{At}

Given these similarities, it is likely we can find a way to define the entries to the matrix e^{At} in terms of e^{t}. Indeed, in some cases, the series converges symbolically. For example, given the lower-triangular A matrix,

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 3 & 0 \end{bmatrix}$$

we find that the approximation sum for e^{At} converges symbolically to

$$\begin{bmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ -t + 3t^2 & 3t & 1 \end{bmatrix}$$

after just 2 iterations of the sum.

However, this nicety is far from universal. Even something that looks as nice as a real and diagonally populated matrix, such as

$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

can end up with a messy series approximation. After just four iterations through a Maple program for our series approximation, we end up with the matrix

$$\begin{bmatrix} 1 - 4t + 8t^2 - \frac{32}{3}t^3 + \frac{32}{3}t^4 & 0 & 0 \\ 0 & 1 + 3t + \frac{9}{2}t^2 + \frac{9}{2}t^3 + \frac{27}{8}t^4 & 0 \\ 0 & 0 & 1 + 7t + \frac{49}{2}t^2 + \frac{343}{6}t^3 + \frac{2401}{24}t^4 \end{bmatrix}$$

which has been simplified by Maple for display.

It seems apparent that the series is unlikely to converge symbolically. However, with some examination, we note that the terms of the series in each position are deeply related to the value in that position in the original matrix. It is also worthwhile to note that the eigenvalues of a diagonal matrix turn out to be the values populating the diagonal: in this case, -4, 3, and 7. We then examine the terms of the aforementioned infinite series presentation of *e*, and it becomes clear that the actual value populating the exact solution matrix is

$$\begin{bmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{7t} \end{bmatrix}$$

which can be generalized, for this nice type of diagonally populated matrix at least, to

$$\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$$

which is very nice indeed.

A special case: Diagonalizable Matrices

We have made note of the nice solution form for a diagonal matrix. Can we generalize this finding further? It turns out that we can. One might recall that when a matrix is diagonalizable, the eigenvalues of the original matrix and the diagonalized matrix are the same. Therefore, we could reasonably expect that the solution to e^{At} for a diagonalizable matrix to be the same as the solution for the associated diagonal matrix. Let's take a look at a more "average" matrix such as

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

For the sake of brevity: The characteristic polynomial for this matrix is $x^3 - 3x^2 + x - 3$; the eigenvalues for this matrix are therefore $\lambda = 3, i$, or $-i$. This seems a bit alarming at first—after all, this is another anonymous matrix with all real entires, and we saw a negative entry in our last example. If we try using the summation definition to find e^{At}, we don't see a simple relation to the original eigenvalues like we did with the diagonal matrix.

However, let's examine the eigenvectors from our previous, nicer (diagonal) matrix. If we place the eigenvectors into a matrix, with row 1 being λ_1's eigenvector and so forth, we have the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In other words—the identity matrix. This suggests that having a diagonal matrix is desirable, so maybe we should try diagonalizing our matrix.

We already know our eigenvalues, as stated above - λ = 3, i, or -i. We have a diagonalizable matrix since we have 3 distinct eigenvalues for our 3-dimensional matrix. Therefore, our D for the diagonalization process will look something like

$$D = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

and the associated P,

$$P = \begin{bmatrix} -i & i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, our diagonalized matrix allows us to write the equality

$$e^{At} = e^{PDP^{-1}t}$$

which could be nicer, supposing we could isolate the diagonal part, as this would enable a very fast calculation of the interior matrix based on our earlier discovered general solution for diagonal matrices.

$$e^{PDP^{-1}t} = Pe^{Dt}P^{-1}$$

Without implying any triviality to the actual calculations involved with computing the above matrix products, we present the solution:

$$\begin{bmatrix} \frac{e^{it}}{2} + \frac{e^{-it}}{2} & \frac{-ie^{it}}{2} + \frac{ie^{-it}}{2} & 0 \\ \frac{ie^{it}}{2} - \frac{ie^{-it}}{2} & \frac{e^{it}}{2} + \frac{e^{-it}}{2} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

Differentiation of solutions to e^{At}

A handy property of e^{nx} is that *d/dx* is always ne^{nx}. One might not immediately expect such a property to carry over to the world of the matrix exponential, but it turns out that in the way we expect,

$$\frac{d}{dt}e^{At} = Ae^{At}$$

which we can prove using the infinite series definition of e^{At}. When we differentiate the *k*th term of the sequence, we note that

$$\frac{d}{dt}\frac{A^k}{(k)!}t^k = k\frac{(A)A^{k-1}}{(k)!}t^{k-1} = A\frac{A^{k-1}}{(k-1)!}t^{k-1}$$

Because the differentiation of the sum is equivalent to the differentiation of the interior expressions,

$$\frac{d}{dt}\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k = \sum_{k=0}^{\infty}\frac{d}{dt}\frac{A^k}{k!}t^k = \sum_{k=1}^{\infty}A\frac{A^{k-1}}{(k-1)!}t^{k-1} = A\sum_{k=1}^{\infty}\frac{A^{k-1}}{(k-1)!}t^{k-1}$$

Because the first term series to be differentiated is always a constant, we are able to shift the summation index, *k,* forward to 1 without affecting the sum because differentiating a constant always yields zero. We can substitute *k'* = *k* - 1 to find that

$$A\sum_{k=1}^{\infty}\frac{A^{k-1}}{(k-1)!}t^{k-1} = A\sum_{k'=0}^{\infty}\frac{A^{k'}}{(k')!}t^{k'} = Ae^At$$

which was to be demonstrated .

Properties of solutions to e^{At}

We have seen before the zero vector, **0**, as a vector where all elements in the vector, however numerous, are zero. A similar notion of zero can be established for matrices. Intuitively, one expects that such a zero matrix would be populated entirely by zeros, and indeed, this is the definition we will use. Likewise, we will use the same bold zero to denote the zero matrix symbolically.

Under certain conditions, the solution to

$$\lim_{t\to\infty} e^{At}$$

will be our zero matrix, **0**. We call such systems *stable* systems. In another facet, the solutions to all components of such systems given by t will decay to 0 over time. When can we expect such systems to be stable? Clearly, the determination of this will rest entirely on our variable A matrix, but how?

We've seen the role the eigenvalues of a matrix play in the solution to e^{At} - elements in the solution matrix follow the pattern $e^{\lambda t}$, so we should examine what values of λ will drive the terms towards zero. We must also be mindful of the fact that our eigenvalues can be imaginary. This being the case, we need to use the definition of the exponential in the complex plane. Specifically,

$$e^{(c+id)} = e^{c}(\cos d + i \sin d)$$

where c is the real component and id is the imaginary component. Using what we know about the exponential function in the real plane–specifically,

$$\lim_{c\to-\infty} e^{c} = 0$$

we can show, since the sine and cosine of d do not approach a large positive or negative value, that

$$\lim_{c\to-\infty} e^{c}(\cos d + i \sin d) = 0(\cos d + i \sin d) = 0$$

which would dictate that any c, and therefore any λ, less than zero will induce a zero solution and therefore a zero entry in the solution matrix for $e^{\lambda t}$ in the position(s) corresponding to that λ .

Thusly, any diagonal or diagonalizable matrix having the property

$$\forall\lambda \in eigenvalues(A) : \lambda < 0$$

will be stable.

It is then intuitively expected that a system will not be stable (will not tend towards the zero matrix as t approaches infinity) if the entries in the solution matrix do not tend towards zero. We need only show a single non-zero entry will exist in the solution matrix if one of the eigenvalues is greater than or equal to zero. For the case of a diagonalizable matrix this is relatively simple - we know the entry $e^{\lambda t}$ will be present at least once in the solution matrix, and as long as λ is a constant greater than zero,

$$\lim_{t\to\infty} e^{\lambda t} = \infty$$

and we will therefore have at least one non-zero entry in the solution matrix.

Proving this in general involves a bit more cleverness. We must utilize the fact that

$$e^{At}\vec{v} = e^{\lambda t}\vec{v}$$

where the vector $\vec{v}$ is the eigenvector associated with the positive λ. From this it is clear that if λ is greater than or equal to zero, $e^{\lambda t}$ will approach something non-zero: indeed, infinity or one. This implies that e^{At} will also approach a non-zero matrix, and is therefore not stable.

Limitations of applicability

There are, however, times where we cannot use these simplifying rules. Let's examine a system of differential equations:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = \alpha x - 2y$$

We see that the A matrix,

$$\begin{bmatrix} 0 & 1 \\ \alpha & -2 \end{bmatrix}$$

has the characteristic polynomial

$$\lambda^2 + 2\lambda - \alpha$$

We are unable to factor this, but we can still find our eigenvalues with the quadratic formula:

$$-2 \pm \frac{\sqrt{4+4\alpha}}{2} = -1 \pm \sqrt{1+\alpha}$$

We must constrain our α such that our eigenvalues are less than zero to ensure the resultant system is stable, as shown in the previous section. However, border cases tend to cause issues. We have already eliminated 0 in our last section, but what of the case where $\alpha = -1$? In this case, our quadratic formula's two answers would be -1 + 0 and -1 - 0, which are clearly equal. Therefore, we would have just one distinct eigenvalue: -1. Because our matrix is two-dimensional, this would imply that our eigenvalue has multiplicity greater than one, which further implies that the matrix would not be diagonalizable for this case. Not having a diagonalizable matrix would prevent us from using the methods developed in the section on the special case of the exponential of diagonalizable matrices to find a solution to e^{At}. Therefore, if we wish to employ the efficient methods we now understand for diagonalizable matrices, the constraint on α would be

$$\alpha < 0, \alpha \neq -1$$

with any real α value in the range producing a stable exponential, e^{At}.

APPENDIX/ACKNOWLEDGEMENTS

From pages 3/4: A Maple program to find the matrix after n terms of the summation:

```
with(LinearAlgebra):
eAt := proc (A, numIters) local k, currentTerm, fullSequence, prevSequence;
fullSequence := 0:
prevSequence := 0:
for k from 0 to numIters do
	currentTerm := A^k*t^k/factorial(k);
	prevSequence := fullSequence;
	fullSequence := prevSequence+currentTerm;
end do;
end proc;
```

From page 6: The simplification of the exponential of the diagonalizable matrix into something more workable,

$$e^{PDP^{-1}t} = Pe^{Dt}P^{-1}$$

is attributable entirely to the text author's cleverness.

From cover and page 2: Plots created using Mac OS X 10.5 Grapher, using example systems of linear differential equations from example sheet "First order homogeneous linear systems of differential equations with constant coefficients" by V. Naibo, Kansas State University. Available via Internet.

A copy of the Maple workbook used in several of the computations in this document is available at http://homepage.mac.com/kiel_oleson/math314exp.mw.

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